FURTHER MATHS

WELCOME TO A Level FURTHER MATHEMATICS AT CARDINAL NEWMAN CATHOLIC SCHOOL

This two-year Edexcel Pearson syllabus is intended for high ability candidates who have achieved, or are likely to achieve, a high grade in the A Level Mathematics examination. The A Level Further Mathematics syllabus enables students to extend the mathematical skills, knowledge and understanding developed in the A Level Mathematics course. The content of the syllabus covers the areas of Extended Pure Mathematics, Mechanics and Statistics, Decision Maths. Knowledge of the whole content of the A Level Mathematics syllabus is assumed.

Any student who wishes to follow this course should have followed the Higher Tier GCSE and obtained at least a grade 7

Year 1 Further Mathematics	
Paper 1: Core Pure Mathematics 1 *Written examination: 1 hour and 30 minutes* *50% of the qualification* *75 marks*	***Content overview*** *Proof, Complex numbers, Matrices, Further algebra and functions, Further calculus, Further vectors*
Paper 2: Further Mathematics Option 1: Decision Maths 1 *Written examination: 1 hour and 30 minutes* *50% of the qualification* *75 marks*	***Content overview*** *2D: Decision Mathematics - Algorithms and graph theory, Algorithms on graphs, Algorithms on graphs II, Critical path analysis, Linear programming*

Year 1 of the A Level is internally assessed in June

Year 2 Further Mathematics	
Paper 1: Core Pure Mathematics 1 *Written examination: 1 hour and 40 minutes* *25% of the qualification* *80 marks*	***Content overview*** *Proof, Complex numbers, Matrices, Further algebra and functions, Further calculus, Further vectors*
Paper 2: Core Pure Mathematics 2 *Written examination: 1 hour and 40 minutes* *25% of the qualification* *80 marks*	***Content overview*** *Complex numbers, Further algebra and functions, Further calculus, Polar coordinates, Hyperbolic functions, Differential equations*
Paper 3: Further Mathematics Option 2: Further Mechanics 1 and Decision Maths 1 *Written examination: 1 hour and 30 minutes* *50% of the qualification* *80 marks*	***Content overview*** *3C: Further Mechanics 1 - Momentum and impulse, Collisions, Centres of mass, Work and energy, Elastic strings and springs* *3D: Decision Mathematics 1 - Algorithms and graph theory, Algorithms on graphs, Algorithms on graphs II, Critical path analysis, Linear programming*

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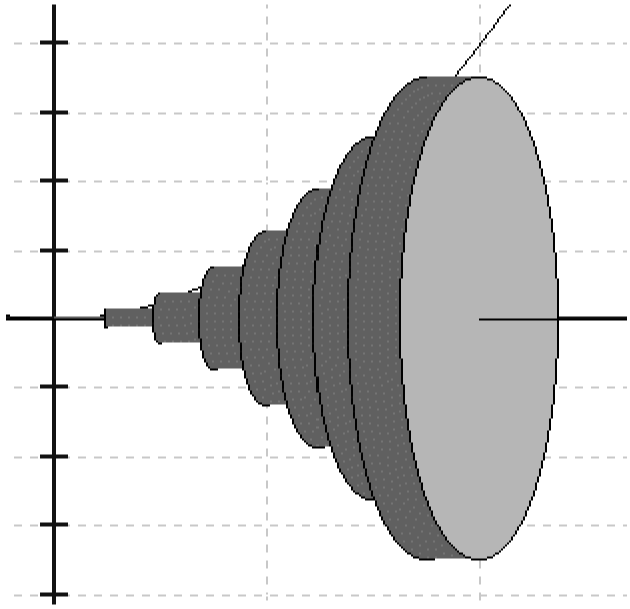
Introduction to A level Further Maths



Over the summer, we would like you to attempt these quick reviews and complete the tasks after carefully reading the lessons and going through the examples and the practice exercises with answers provided.

They cover all the basic skill you ought to be familiar with before you start A-level Further Maths, includes:

- **Quadratic Inequalities**
- **Sketching cubic and reciprocal graphs**
- **3D shapes**
    - **Surface Area**
    - **Volume**
- **Trigonometry**
- **Basic knowledge of Complex Numbers**

It is vital that you complete all the tasks and have them with you on the first day back in September. Within the first week when you return, we will test you on these topics (baseline test). Some of the questions here will be used in the test. Then we will go over all the questions and answers in order for you to do the corrections.

WORK TO BE COMPLETED BY SEPTEMBER (11 tasks):

Task 1 from Page 5, Task 2 from page 13 Task 3 from page 13 Task 4 from page 14

Task 5 from page 17 Task 6 from page 21 Task 7 from page 25 Task 8 from page 28

Task 9 from page 32 Task 10 from page 34 Task 11 from page 37

At the end of each topic, you also have the Challenge question for you to have a go.

Good luck!

1.1 You can manipulate inequalities to solve them.

In C1 you learnt how to solve simple quadratic inequalities by rearranging them. The inequality sign can be treated like an equals sign as long as you do not divide or multiply both sides of the expression by a negative number.

There are three steps to solving inequalities.

Example 1

Solve $2x^2 < x + 3$

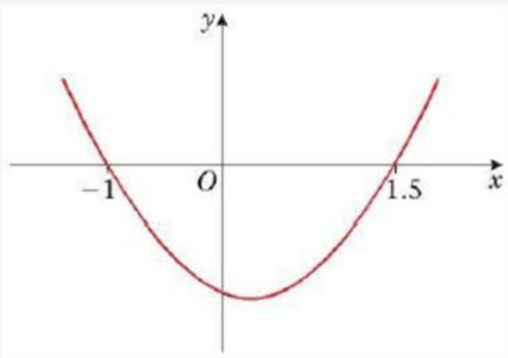
$2x^2 - x - 3 < 0$

$2x^2 - x - 3 = 0$

So $(2x - 3)(x + 1) = 0$

So the critical values are $x = \frac{3}{2}$ or -1

A sketch of $y = 2x^2 - x - 3$ gives



So the solution to $y = 2x^2 - x - 3 < 0$

is when $-1 < x < 1.5$

Step 1 is to find the critical values. Rearrange the expression and then replace the inequality symbol with an equals sign and solve.

Step 2 is to draw a sketch, or use a table of values to determine which sets of values satisfy the inequality.

Step 3 is to write down the answer by using the graph to interpret the inequality.

Example 2

Solve the inequality $\frac{x^2}{x - 2} < x + 1, x \neq 2$

Multiply both sides by $(x - 2)^2$

$(x - 2)^2 \times \frac{x^2}{x - 2} < (x - 2)^2 \times (x + 1)$

A natural first step would be to multiply both sides by $(x - 2)$ but we cannot be sure that this is positive. A simple solution is to multiply both sides of the inequality by $(x - 2)^2$ as this will always be positive.

$$(x-2)^{\cancel{2}} \times \frac{x^2}{\cancel{(x-2)}} < (x-2)^2 \times (x+1)$$

Do **not** aim to multiply out but cancel, collect terms on one side and **factorise**.

$(x-2)x^2 - (x+1)(x-2)^2 < 0$

$(x-2)(x^2 - (x+1)(x-2)) < 0$

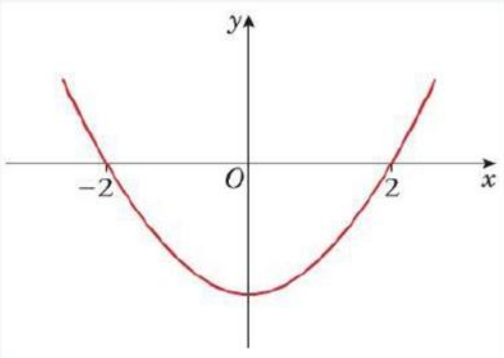
$(x-2)(x^2 - x^2 + x + 2) < 0$

or $(x-2)(x+2) < 0$

Critical values $x = \pm 2$

The sketch of $y = (x-2)(x+2)$ is

Now the problem is similar to those seen in C1. You find the critical values, draw a sketch and write down the answers.



The solution to $y = (x-2)(x+2) < 0$ is $-2 < x < 2$

The same approach can be used in more complicated situations.

Example 3

Solve the inequality $\frac{x}{x+1} \leqslant \frac{2}{x+3}$ $\quad x \neq -1, x \neq -3$

This time multiply both sides by

$(x+1)^2(x+3)^2$

In order to remove the fractions and guarantee that you are not multiplying by a negative quantity, use $(x+1)^2(x+3)^2$.

So

$$(x+1)^{\cancel{2}}(x+3)^2 \times \frac{x}{\cancel{(x+1)}} \leqslant \frac{2}{\cancel{(x+3)}} \times (x+1)^2(x+3)^{\cancel{2}}$$

Cancel terms on each side.

$x(x+1)(x+3)^2 - 2(x+1)^2(x+3) \leqslant 0$

Collect terms on LHS.

$(x+1)(x+3)(x(x+3) - 2(x+1)) \leqslant 0$

Factorise as much as possible.

$(x+1)(x+3)(x^2 + x - 2) \leqslant 0$

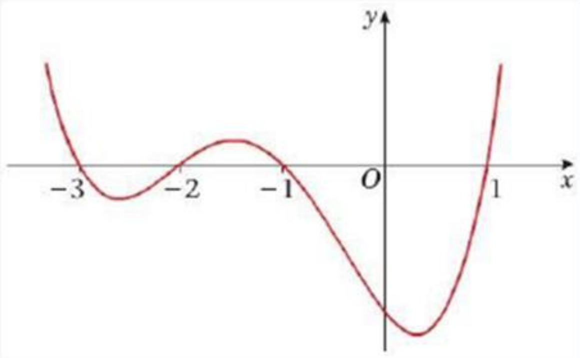
$(x+1)(x+3)(x+2)(x-1) \leqslant 0$

To find the critical values you need to solve $(x+1)(x+3)(x+2)(x-1) = 0$.

So the critical values are:

$x = -1, -3, -2$ or 1

A sketch of $y = (x + 1)(x + 3)(x + 2)(x - 1)$ is



So the solution to

$y = (x + 1)(x + 3)(x + 2)(x - 1) \leqslant 0$

is $-3 < x \leqslant -2$ or $-1 < x \leqslant 1$

The curve $y = (x + 1)(x + 3)(x + 2)(x - 1)$ is essentially an x^4 curve, so it starts in top left and ends in top right and passes through $x = -1, -3, -2$ and 1. The exact shape does not matter.

If the question has a $\leqslant$ as opposed to a $<$ then you must check carefully at the end whether the 'ends' of the set of values are valid. Since $x \neq -1$ or -3 these values must not be included, hence $<$ not $\leqslant$ is used.

TO DO (Task 1):

Exercise 1A

Solve the following inequalities

1 $x^2 < 5x + 6$

2 $x(x + 1) \geqslant 6$

3 $\dfrac{2}{x^2 + 1} > 1$

4 $\dfrac{2}{x^2 - 1} > 1$

5 $\dfrac{x}{x - 1} \leqslant 2x \quad x \neq 1$

6 $\dfrac{3}{x + 1} < \dfrac{2}{x}$

7 $\dfrac{3}{(x + 1)(x - 1)} < 1$

8 $\dfrac{2}{x^2} \geqslant \dfrac{3}{(x + 1)(x - 2)}$

9 $\dfrac{2}{x - 4} < 3$

10 $\dfrac{3}{x + 2} > \dfrac{1}{x - 5}$

11 $\dfrac{3x^2 + 5}{x + 5} > 1$

12 $\dfrac{3x}{x - 2} > x$

13 $\dfrac{1 + x}{1 - x} > \dfrac{2 - x}{2 + x}$

14 $\dfrac{x^2 + 7x + 10}{x + 1} > 2x + 7$

15 **a** $\dfrac{x + 1}{x^2} > 6$ **b** $\dfrac{x^2}{x + 1} > \dfrac{1}{6}$

CHALLENGE

1 Find the set of values of x for which

$$16x \leqslant 8x^2 - x^3.$$

E

2 Find the set of values of x for which

$$\frac{2}{x-2} < \frac{1}{x+1}.$$

E

3 Find the set of values of x for which

$$\frac{x^2}{x-2} > 2x.$$

E

4 Find the set of values of x for which

$$\frac{x^2-12}{x} > 1.$$

E

5 Find the set of values of x for which

$$2x - 5 > \frac{3}{x}.$$

E

6 Given that k is a constant and that $k > 0$, find, in terms of k, the set of values of x for which $\frac{x+k}{x+4k} > \frac{k}{x}$.

E

4.1 You can sketch cubic curves of the form $y = ax^3 + bx^2 + cx + d$

Example 1

Sketch the curve with the equation $y = (x - 2)(x - 1)(x + 1)$

$0 = (x - 2)(x - 1)(x + 1)$

Put $y = 0$ and solve for x to find the roots (the points where the curve crosses the x-axis).

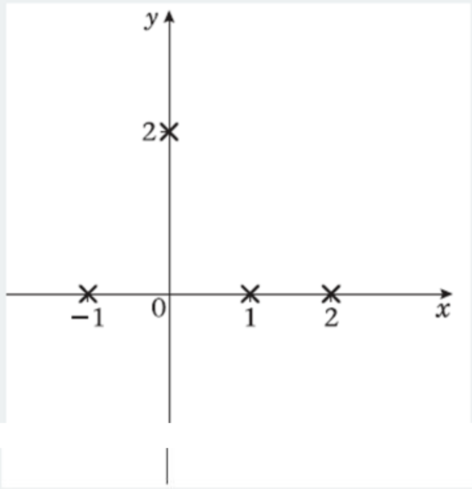
So $x = 2$ or $x = 1$ or $x = -1$

So the curve crosses the x-axis at $(2, 0)$ $(1, 0)$ and $(-1, 0)$.

When $x = 0$, $y = -2 \times -1 \times 1 = 2$

Put $x = 0$ to find where the curve crosses the y-axis.

So the curve crosses the y-axis at $(0, 2)$.



When x is large and positive, y is large and positive.

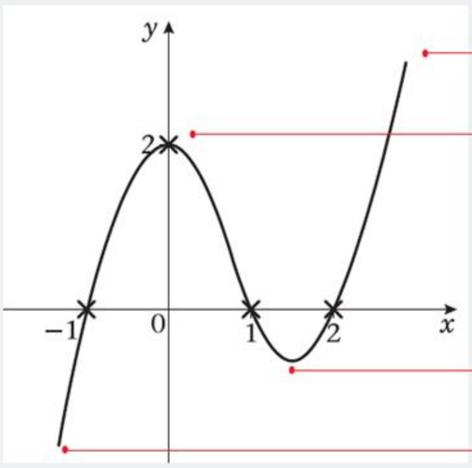
Check what happens to y for large positive and negative values of x.

When x is large and negative, y is large and negative.

You can write this as

$x \to \infty,\ y \to \infty$

$x \to -\infty,\ y \to -\infty$



$x \to \infty,\ y \to \infty$

This is called a maximum point because the gradient changes from +ve to 0 to −ve.

This is called a minimum point because the gradient changes from −ve to 0 to +ve.

$x \to -\infty,\ y \to -\infty$

In your exam you will not be expected to work out the coordinates of the maximum or minimum points without further work, but you should mark points where the curve meets the axes.

Example 2

Sketch the curves with the following equations and show the points where they cross the coordinate axes.

a $y = (x - 2)(1 - x)(1 + x)$ **b** $y = x(x + 1)(x + 2)$

a $0 = (x - 2)(1 - x)(1 + x)$

So $x = 2$, $x = 1$ or $x = -1$

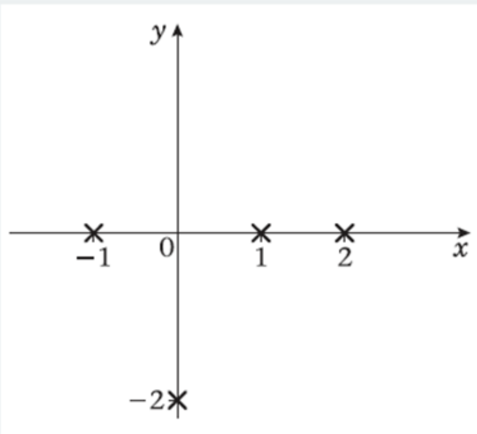
So the curve crosses the x-axis at $(2, 0)$, $(1, 0)$ and $(-1, 0)$.

Put $y = 0$ and solve for x.

When $x = 0$, $y = -2 \times 1 \times 1 = -2$

So the curve crosses the y-axis at $(0, -2)$.

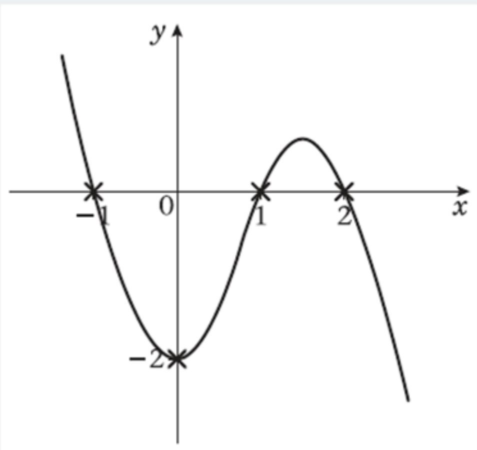
Find the value of y when $x = 0$.



$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$

Check what happens to y for large positive and negative values of x.



Notice that this curve is a reflection in the x-axis of the curve in Example 1.

b $y = x(x + 1)(x + 2)$

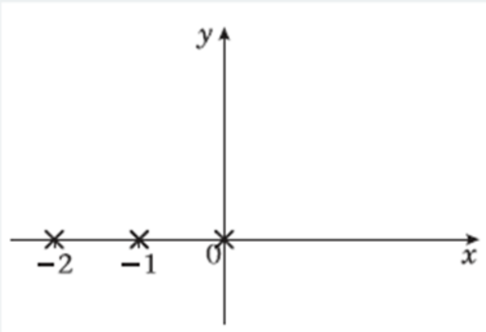
$0 = x(x + 1)(x + 2)$

So $x = 0$, $x = -1$ or $x = -2$

Put $y = 0$ and solve for x.

So the curve crosses the x-axis at $(0, 0)$, $(-1, 0)$ and $(-2, 0)$.

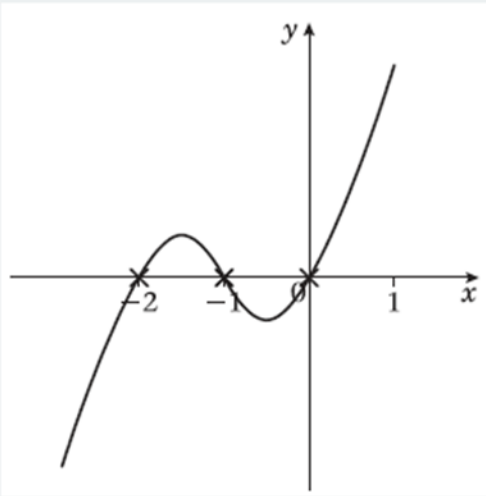
So the curve crosses the y-axis at $(0, 0)$.



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



Example 3

Sketch the following curves.

a $y = (x - 1)^2(x + 1)$

b $y = x^3 - 2x^2 - 3x$

a $y = (x - 1)^2(x + 1)$

$0 = (x - 1)^2(x + 1)$

So $x = 1$ or $x = -1$.

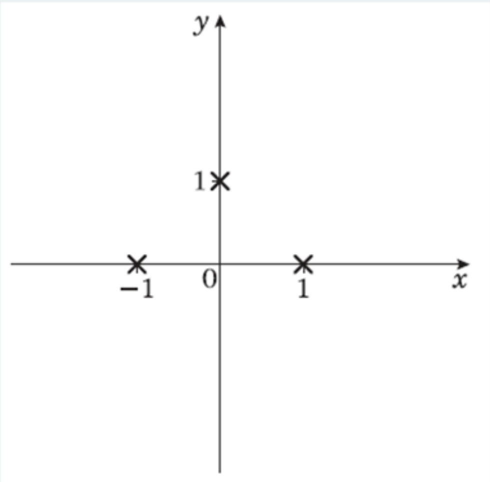
Put $y = 0$ and solve for x.

So the curve crosses the x-axis at $(1, 0)$ and $(-1, 0)$.

When $x = 0$ $y = (-1)^2 \times 1 = 1$

Find the value of y when $x = 0$.

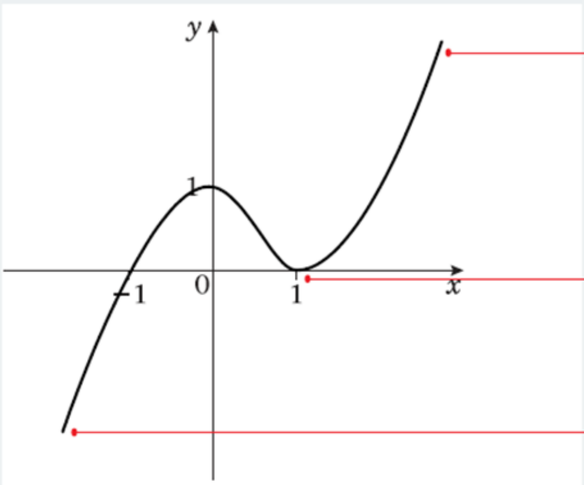
So the curve crosses the y-axis at $(0, 1)$.



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

Check what happens to y for large positive and negative values of x.



$x \to \infty, y \to \infty$

$x = 1$ is a 'double' root.

$x \to -\infty, y \to -\infty$

TO DO (Task 2):

Exercise 4A

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x - 3)(x - 2)(x + 1)$ **b** $y = (x - 1)(x + 2)(x + 3)$

c $y = (x + 1)(x + 2)(x + 3)$ **d** $y = (x + 1)(1 - x)(x + 3)$

e $y = (x - 2)(x - 3)(4 - x)$ **f** $y = x(x - 2)(x + 1)$

g $y = x(x + 1)(x - 1)$ **h** $y = x(x + 1)(1 - x)$

i $y = (x - 2)(2x - 1)(2x + 1)$ **j** $y = x(2x - 1)(x + 3)$

2 Sketch the curves with the following equations:

a $y = (x + 1)^2(x - 1)$ **b** $y = (x + 2)(x - 1)^2$

c $y = (2 - x)(x + 1)^2$ **d** $y = (x - 2)(x + 1)^2$

e $y = x^2(x + 2)$ **f** $y = (x - 1)^2x$

g $y = (1 - x)^2(3 + x)$ **h** $y = (x - 1)^2(3 - x)$

i $y = x^2(2 - x)$ **j** $y = x^2(x - 2)$

3 Factorise the following equations and then sketch the curves:

a $y = x^3 + x^2 - 2x$ **b** $y = x^3 + 5x^2 + 4x$

c $y = x^3 + 2x^2 + x$ **d** $y = 3x + 2x^2 - x^3$

e $y = x^3 - x^2$ **f** $y = x - x^3$

g $y = 12x^3 - 3x$ **h** $y = x^3 - x^2 - 2x$

i $y = x^3 - 9x$ **j** $y = x^3 - 9x^2$

4.2 **You need to be able to sketch and interpret graphs of cubic functions of the form $y = x^3$.**

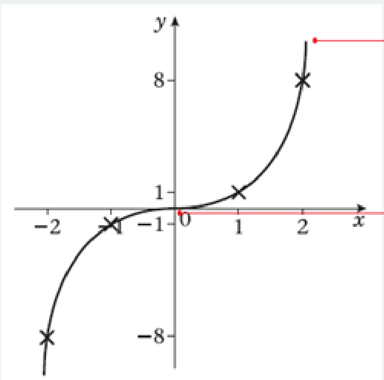
Example 4

Sketch the curve with equation $y = x^3$.

$0 = x^3$

So the curve crosses both axes at (0, 0).

x	-2	-1	0	1	2
$y = x^3$	-8	-1	0	1	8



Put $y = 0$ and solve for x.

As the curve passes the axes at only one point, find its shape by plotting a few points.

Notice that as x increases, y increases rapidly.

The curve is 'flat' at (0, 0). This point is called a point of inflexion. The gradient is positive just before (0, 0) and positive just after (0, 0).

Notice that the shape of this curve is the same as the curve with equation $y = (x + 1)^3$, which is shown in Example 5.

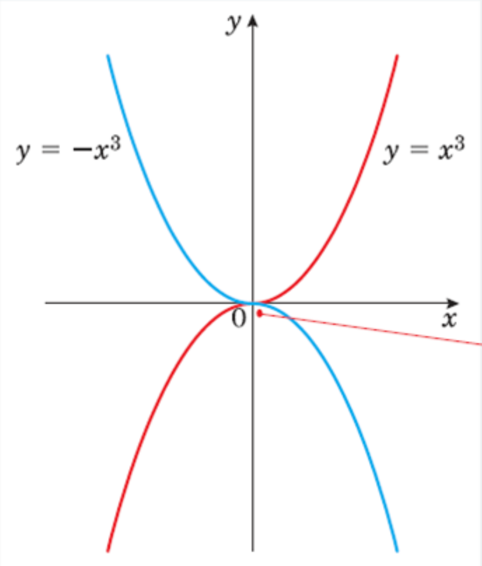
Example 5

Sketch the curve with equations:

a $y = -x^3$ **b** $y = (x + 1)^3$ **c** $y = (3 - x)^3$

Show their positions relative to the curve with equation $y = x^3$.

a $y = -x^3$



You do not need to plot any points. It is quicker if you realise the curve $y = -x^3$ is a reflection in the x-axis of the curve $y = x^3$. You can check this by looking at the values used to sketch $y = x^3$. So, for example, $x = 2$ will now correspond to $y = -8$ on the curve $y = -x^3$.

The curve is still flat at (0, 0).

b $y = (x + 1)^3$

$0 = (x + 1)^3$

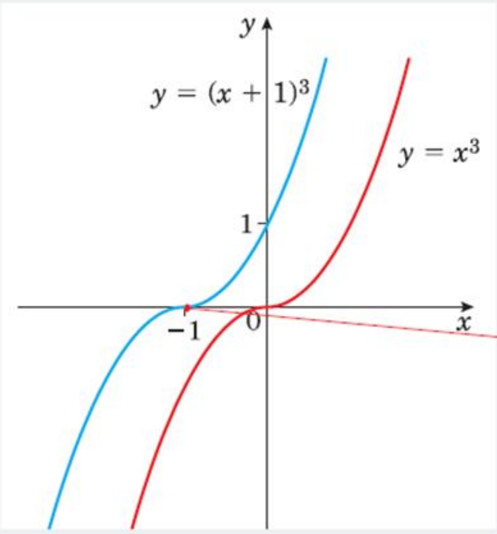
So $x = -1$

So the curve crosses the x-axis at $(-1, 0)$.

When $x = 0$, $y = 1^3 = 1$

So the curve crosses the y-axis at $(0, 1)$.

Put $y = 0$ to find where the curve crosses the x-axis.
Put $x = 0$ to find where the curve crosses the y-axis.



The curve has the same shape as $y = x^3$.

You do not need to do any working if you realise the curve $y = (x + 1)^3$ is a translation of -1 along the x-axis of the curve $y = x^3$.

The point of inflexion is at $(-1, 0)$.

TO DO (Task 3)

Exercise 4B

1 Sketch the following curves and show their positions relative to the curve $y = x^3$:

a $y = (x - 2)^3$ **b** $y = (2 - x)^3$ **c** $y = (x - 1)^3$

d $y = (x + 2)^3$ **e** $y = -(x + 2)^3$

2 Sketch the following and indicate the coordinates of the points where the curves cross the axes:

a $y = (x + 3)^3$ **b** $y = (x - 3)^3$ **c** $y = (1 - x)^3$

d $y = -(x - 2)^3$ **e** $y = -(x - \frac{1}{2})^3$

4.3 You need to be able to sketch the reciprocal function $y = \dfrac{k}{x}$ where k is a constant.

Example 6

Sketch the curve $y = \dfrac{1}{x}$ and its asymptotes.

$y = \dfrac{1}{x}$

When $x = 0$, y is not defined.

When $y = 0$, x is not defined.

$x \to +\infty$, $y \to 0$

$x \to -\infty$, $y \to 0$

$y \to +\infty$, $x \to 0$

$y \to -\infty$, $x \to 0$

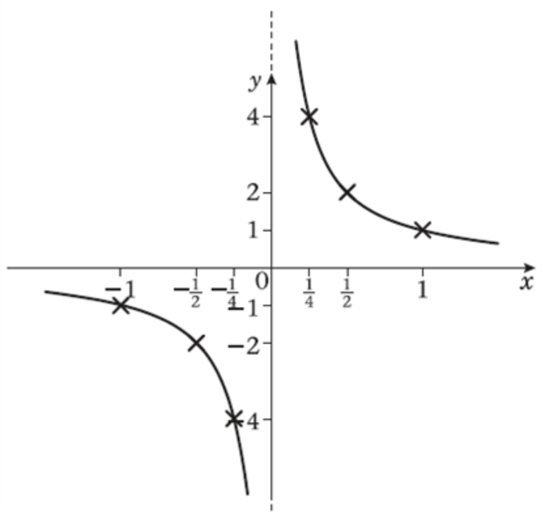
The curve does not cross the axes.

The curve tends towards the x-axis when x is large and positive or large and negative. The x-axis is a horizontal asymptote.

The curve tends towards the y-axis when y is large and positive or large and negative. The y-axis is a vertical asymptote.

$y = \dfrac{1}{x}$

x	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1
$y = \frac{1}{x}$	-1	-2	-4	4	2	1



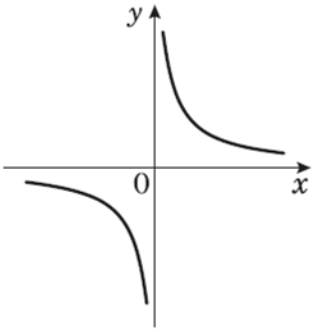
The curve does not cross the x-axis or y-axis. You need to plot some points.

You can draw a dashed line to indicate an asymptote. (In this case the asymptotes are the axes, but see Example 11.)

- **The curves with equations $y = \frac{k}{x}$ fall into two categories:**

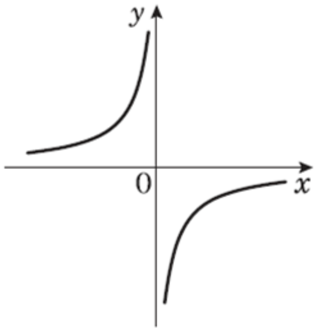
Type 1

$y = \frac{k}{x},\ k > 0$



Type 2

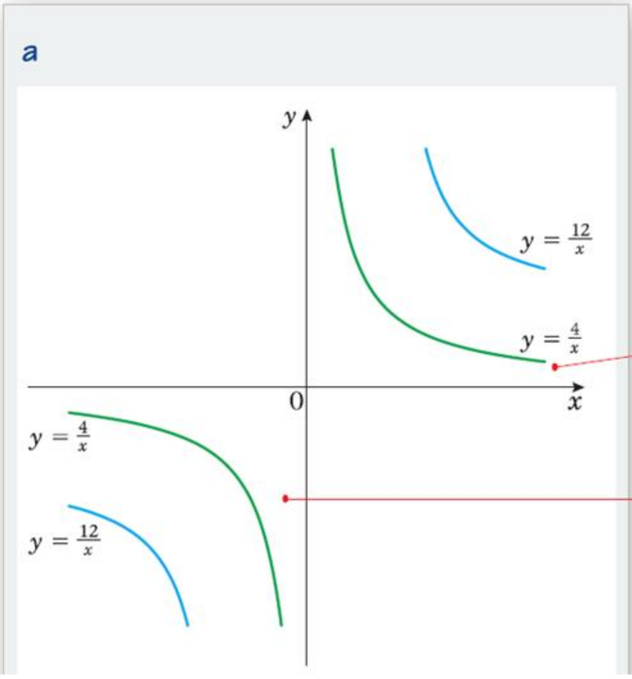
$y = \frac{k}{x},\ k < 0$



Example 7

Sketch on the same diagram:

a $y = \frac{4}{x}$ and $y = \frac{12}{x}$ **b** $y = -\frac{1}{x}$ and $y = -\frac{3}{x}$



The shape of these curves will be Type 1.

In this quadrant, $x > 0$
so for any values of x: $\frac{12}{x} > \frac{4}{x}$

In this quadrant, $x < 0$
so for any values of x: $\frac{12}{x} < \frac{4}{x}$

TO DO (Task 4)

Exercise 4C

Use a separate diagram to sketch each pair of graphs.

1 $y = \frac{2}{x}$ and $y = \frac{4}{x}$

2 $y = \frac{2}{x}$ and $y = -\frac{2}{x}$

3 $y = -\frac{4}{x}$ and $y = -\frac{2}{x}$

4 $y = \frac{3}{x}$ and $y = \frac{8}{x}$

5 $y = -\frac{3}{x}$ and $y = -\frac{8}{x}$

4.4 You can sketch curves of functions to show points of intersection and solutions to equations.

Example 8

a On the same diagram sketch the curves with equations $y = x(x - 3)$ and $y = x^2(1 - x)$.

b Find the coordinates of the point of intersection.

a $y = x(x - 3)$

$0 = x(x - 3)$

Put $y = 0$ and solve for x.

So $x = 0$ or $x = 3$.

So the curve crosses the x-axis at $(0, 0)$ and $(3, 0)$.

$y = x^2(1 - x)$

$0 = x^2(1 - x)$

Put $y = 0$ and solve for x to find where the curve crosses the x-axis.

So $x = 0$ or $x = 1$.

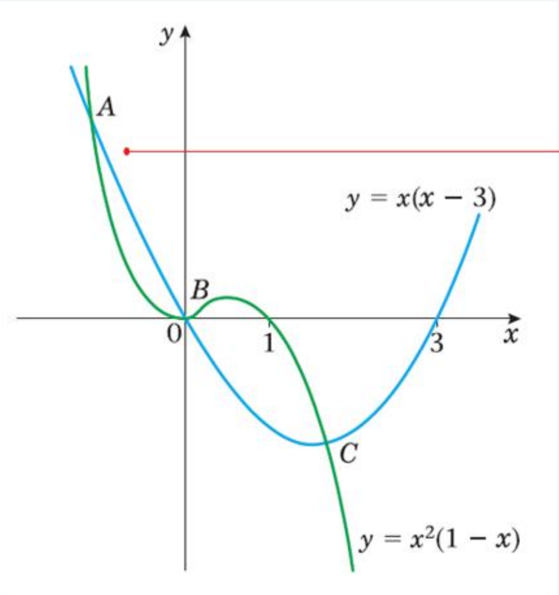
So the curve crosses the x-axis at $(0, 0)$ or $(1, 0)$.

The curve crosses the y-axis at $(0, 0)$.

$x \to \infty,\ y \to -\infty$

$x \to -\infty,\ y \to +\infty$

Check what happens to y for large positive and negative values of x.



A cubic curve is always steeper than a quadratic curve, so it will cross over somewhere on this side of the y-axis.

b From the graph there are three points where the curves cross, labelled A, B and C. The x-coordinates are given by the solutions to the equation.

$$x(x-3) = x^2(1-x)$$

$$x^2 - 3x = x^2 - x^3$$ Multiply out brackets (see Section 1.3).

$$x^3 - 3x = 0$$ Collect terms on one side.

$$x(x^2 - 3) = 0$$ Factorise.

$$x(x-\sqrt{3})(x+\sqrt{3}) = 0$$ Factorise using a difference of 2 squares.

So $x = -\sqrt{3}, 0, \sqrt{3}$

You can use the equation $y = x^2(1-x)$ to find the y-coordinates.

So the point where x is negative is $A\,(-\sqrt{3}, 3[1+\sqrt{3}])$, B is $(0, 0)$ and C is the point $(\sqrt{3}, 3[1-\sqrt{3}])$.

Example 9

a On the same diagram sketch the curves with equations $y = x^2(x-1)$ and $y = \dfrac{2}{x}$.

b Explain how your sketch shows that there are two solutions to the equation $x^2(x-1) - \dfrac{2}{x} = 0$.

a

$$y = x^2(x-1)$$

$$0 = x^2(x-1)$$ Put $y = 0$ and solve for x.

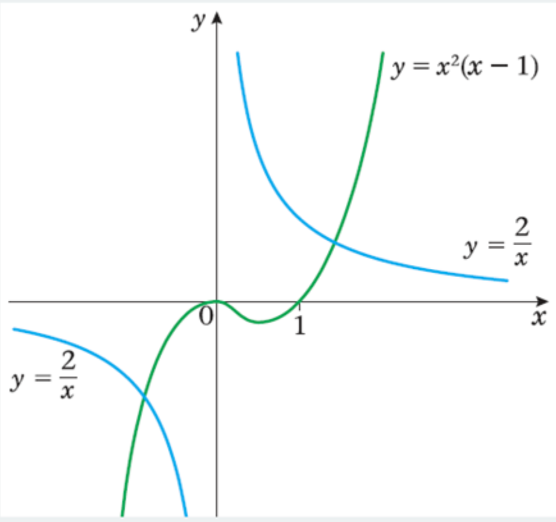
So $x = 0$ or $x = 1$.

So the curve crosses the x-axis at $(0, 0)$ and $(1, 0)$.

The curve crosses the y-axis at $(0, 0)$.

$$x \to \infty,\ y \to \infty$$
$$x \to -\infty,\ y \to -\infty$$

Check what happens to y for large positive and negative values of x.



b From the sketch there are only two points of intersection of the curves. This means there are only two values of x where

$$x^2(x-1) = \frac{2}{x}$$

or $$x^2(x-1) - \frac{2}{x} = 0$$

So this equation has two solutions.

TO DO (Task 5):

Exercise 4D

1 In each case:

i sketch the two curves on the same axes

ii state the number of points of intersection

iii write down a suitable equation which would give the x-coordinates of these points. (You are not required to solve this equation.)

a $y = x^2$, $y = x(x^2 - 1)$

b $y = x(x + 2)$, $y = -\dfrac{3}{x}$

c $y = x^2$, $y = (x + 1)(x - 1)^2$

d $y = x^2(1 - x)$, $y = -\dfrac{2}{x}$

e $y = x(x - 4)$, $y = \dfrac{1}{x}$

f $y = x(x - 4)$, $y = -\dfrac{1}{x}$

g $y = x(x - 4)$, $y = (x - 2)^3$

h $y = -x^3$, $y = -\dfrac{2}{x}$

i $y = -x^3$, $y = x^2$

j $y = -x^3$, $y = -x(x + 2)$

Hint: In question 1f, check the point $x = 2$ in both curves.

2 **a** On the same axes sketch the curves given by $y = x^2(x - 4)$ and $y = x(4 - x)$.

b Find the coordinates of the points of intersection.

3 **a** On the same axes sketch the curves given by $y = x(2x + 5)$ and $y = x(1 + x)^2$

b Find the coordinates of the points of intersection.

4 **a** On the same axes sketch the curves given by $y = (x - 1)^3$ and $y = (x - 1)(1 + x)$.

b Find the coordinates of the points of intersection.

5 **a** On the same axes sketch the curves given by $y = x^2$ and $y = -\frac{27}{x}$.

b Find the coordinates of the point of intersection.

6 **a** On the same axes sketch the curves given by $y = x^2 - 2x$ and $y = x(x - 2)(x - 3)$.

b Find the coordinates of the point of intersection.

7 **a** On the same axes sketch the curves given by $y = x^2(x - 3)$ and $y = \frac{2}{x}$.

b Explain how your sketch shows that there are only two solutions to the equation $x^3(x - 3) = 2$.

8 **a** On the same axes sketch the curves given by $y = (x + 1)^3$ and $y = 3x(x - 1)$.

b Explain how your sketch shows that there is only one solution to the equation $x^3 + 6x + 1 = 0$.

9 **a** On the same axes sketch the curves given by $y = \frac{1}{x}$ and $y = -x(x - 1)^2$.

b Explain how your sketch shows that there are no solutions to the equation $1 + x^2(x - 1)^2 = 0$.

10 **a** On the same axes sketch the curves given by $y = 1 - 4x^2$ and $y = x(x - 2)^2$.

b State, with a reason, the number of solutions to the equation $x^3 + 4x - 1 = 0$.

11 **a** On the same axes sketch the curve $y = x^3 - 3x^2 - 4x$ and the line $y = 6x$.

b Find the coordinates of the points of intersection.

12 **a** On the same axes sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$.

b Find the coordinates of the points of intersection.

13 **a** On the same axes sketch the curves with equations $y = (x - 2)(x + 2)^2$ and $y = -x^2 - 8$.

b Find the coordinates of the points of intersection.

CHALLENGE:

1 **a** On the same axes sketch the graphs of $y = x^2(x - 2)$ and $y = 2x - x^2$.

b By solving a suitable equation find the points of intersection of the two graphs.

2 **a** On the same axes sketch the curves with equations $y = \frac{6}{x}$ and $y = 1 + x$.

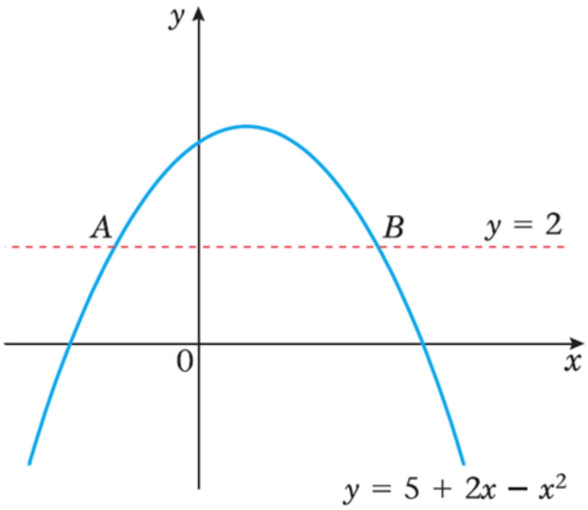
b The curves intersect at the points A and B. Find the coordinates of A and B.

c The curve C with equation $y = x^2 + px + q$, where p and q are integers, passes through A and B. Find the values of p and q.

d Add C to your sketch.

3

The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.

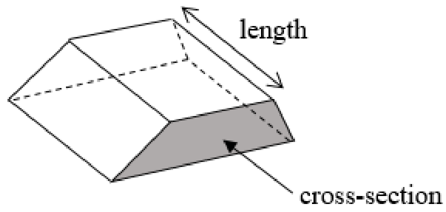


Find the x-coordinates of A and B.

Volume and surface area of 3D shapes

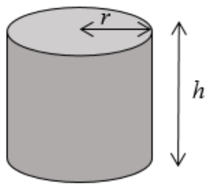
A LEVEL LINKS
Scheme of work: 6b. Gradients, tangents, normals, maxima and minima

Key points

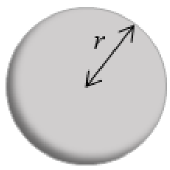


- Volume of a prism = cross-sectional area × length.
- The surface area of a 3D shape is the total area of all its faces.

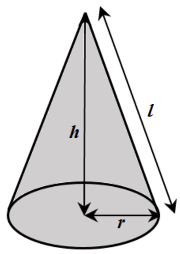
- Volume of a pyramid = $\frac{1}{3}$ × area of base × vertical height.



- Volume of a cylinder = $\pi r^2 h$
- Total surface area of a cylinder = $2\pi r^2 + 2\pi rh$



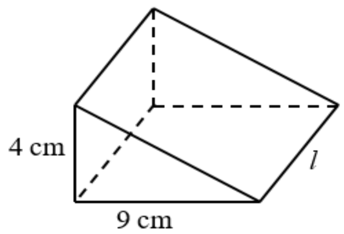
- Volume of a sphere = $\frac{4}{3}\pi r^3$
- Surface area of a sphere = $4\pi r^2$



- Volume of a cone = $\frac{1}{3}\pi r^2 h$
- Total surface area of a cone = $\pi rl + \pi r^2$

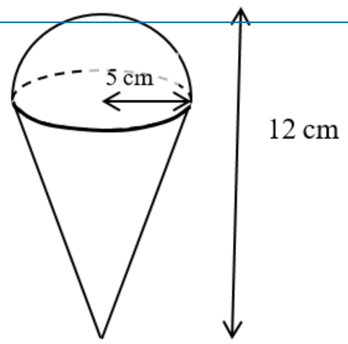
Examples

Example 1 The triangular prism has volume 504 cm^3.
Work out its length.



$V = \frac{1}{2}bhl$	**1** Write out the formula for the volume of a triangular prism.
$504 = \frac{1}{2} \times 9 \times 4 \times l$	**2** Substitute known values into the formula.
$504 = 18 \times l$	**3** Simplify
$l = 504 \div 18$ $= 28$ cm	**4** Rearrange to work out l. **5** Remember the units.

Example 2 Calculate the volume of the 3D solid.
Give your answer in terms of π.

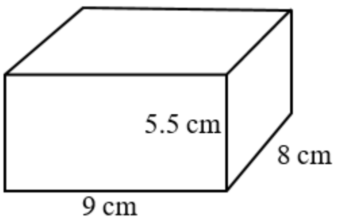


Total volume = volume of hemisphere + Volume of cone $= \frac{1}{2}$ of $\frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$	**1** The solid is made up of a hemisphere radius 5 cm and a cone with radius 5 cm and height $12 - 5 = 7$ cm.
Total volume $= \frac{1}{2} \times \frac{4}{3} \times \pi \times 5^3 + \frac{1}{3} \times \pi \times 5^2 \times 7$	**2** Substitute the measurements into the formula for the total volume.
$= \frac{425}{3}\pi$ cm^3	**3** Remember the units.

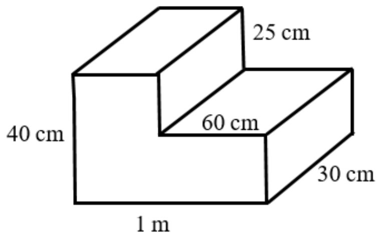
TO DO (Task 6):

1 Work out the volume of each solid.
Leave your answers in terms of π where appropriate.

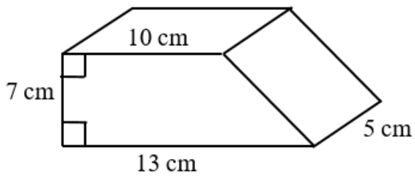
a



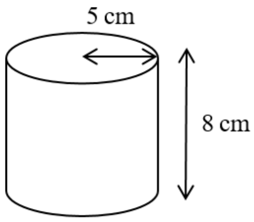
b



c



d



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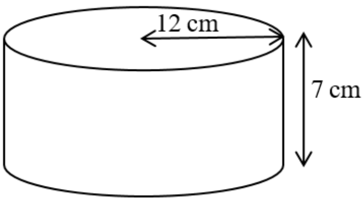
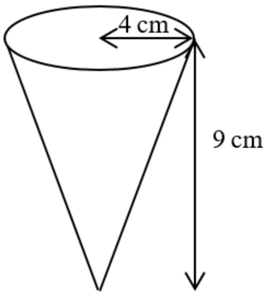


f a sphere with radius 7 cm

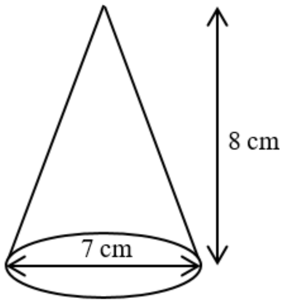
g a sphere with diameter 9 cm

h a hemisphere with radius 3 cm

i

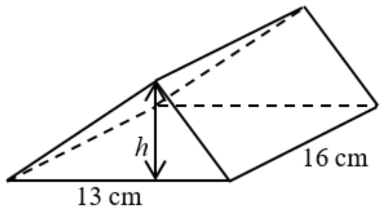


j



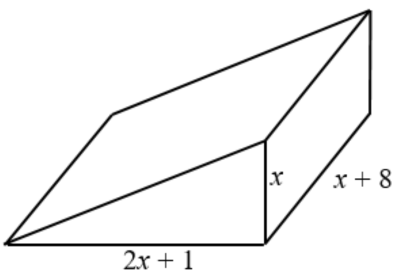
2 A cuboid has width 9.5 cm, height 8 cm and volume 1292 cm^3.
Work out its length.

3 The triangular prism has volume 1768 cm^3.
Work out its height.

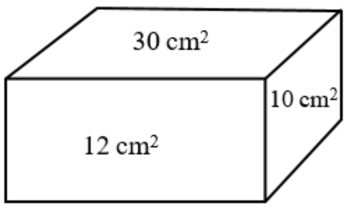


<u>CHALLENGE</u>

4 The diagram shows a solid triangular prism.
All the measurements are in centimetres.
The volume of the prism is V cm^3.
Find a formula for V in terms of x.
Give your answer in simplified form.



5 The diagram shows the area of each of three faces of a cuboid.
The length of each edge of the cuboid is a whole number of centimetres.
Work out the volume of the cuboid.



6 The diagram shows a large catering size tin of beans in the shape of a cylinder.

The tin has a radius of 8 cm and a height of 15 cm.

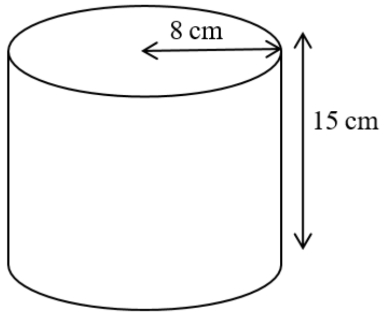
A company wants to make a new size of tin.

The new tin will have a radius of 6.7 cm.

It will have the same volume as the large tin.

Calculate the height of the new tin.

Give your answer correct to one decimal place.



7 The diagram shows a sphere and a solid cylinder.

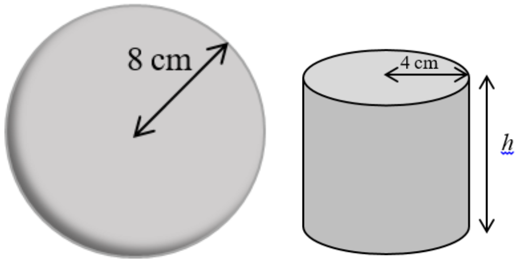
The sphere has radius 8 cm.

The solid cylinder has a base radius of 4 cm and a height of h cm.

The total surface area of the cylinder is half the total surface area of the sphere.

Work out the ratio of the volume of the sphere to the volume of the cylinder.

Give your answer in its simplest form.

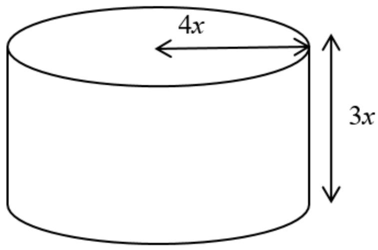


8 The diagram shows a solid metal cylinder.

The cylinder has base radius $4x$ and height $3x$.

The cylinder is melted down and made into a sphere of radius r.

Find an expression for r in terms of x.

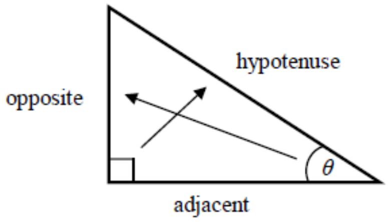


Trigonometry in right-angled triangles

A LEVEL LINKS
Scheme of work: 4a. Trigonometric ratios and graphs

Key points

- In a right-angled triangle:
  - the side opposite the right angle is called the hypotenuse
  - the side opposite the angle θ is called the opposite
  - the side next to the angle θ is called the adjacent.



- In a right-angled triangle:
  - the ratio of the opposite side to the hypotenuse is the sine of angle θ, $\sin\theta = \frac{\text{opp}}{\text{hyp}}$
  - the ratio of the adjacent side to the hypotenuse is the cosine of angle θ, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$
  - the ratio of the opposite side to the adjacent side is the tangent of angle θ, $\tan\theta = \frac{\text{opp}}{\text{adj}}$

- If the lengths of two sides of a right-angled triangle are given, you can find a missing angle using the inverse trigonometric functions: $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

- The sine, cosine and tangent of some angles may be written exactly.

	0	**30°**	**45°**	**60°**	**90°**
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

2.6 You need to be able to use the sine rule, the cosine rule, the trigonometric ratios sin, cos and tan, and Pythagoras' theorem to solve problems.

In triangle work involving trigonometric calculations, the following strategy might help you.

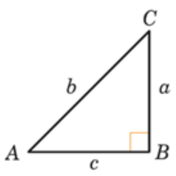
- When the triangle is right-angled or isosceles it is better to use sine, cosine, tangent or Pythagoras' theorem.

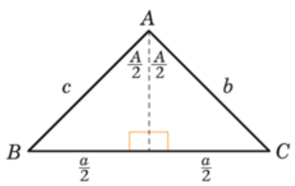
$\sin A = \frac{a}{b}$

$\cos A = \frac{c}{b}$

$\tan A = \frac{a}{c}$

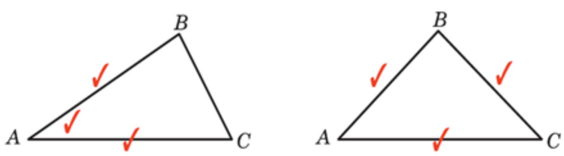
$a^2 + c^2 = b^2$





The line of symmetry produces two right-angled triangles.

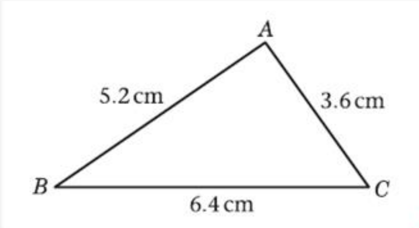
- Use the **cosine rule** when you are given either **two sides and the angle between them** or **three sides**.



- For other combinations of given data, use the **sine rule**.
- When you have used the cosine rule once, it is generally better not to use it again, as the cosine rule involves more calculations and so may introduce more rounding errors.

Example 14

In $\triangle ABC$, $AB = 5.2$ cm, $BC = 6.4$ cm and $AC = 3.6$ cm.
Calculate the angles of the triangle.



$$\cos A = \frac{3.6^2 + 5.2^2 - 6.4^2}{2 \times 3.6 \times 5.2}$$

$$= -0.02564\ldots$$

So $A = 91.5°$ (3 s.f.)

$$\sin C = \frac{5.2 \sin A}{6.4}$$

So $C = 54.3°$

and $B = 34.2°$

It is always better to work out the largest angle first, if you have a choice.

Here that is angle A, so use the cosine rule $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Negative sign indicates an obtuse angle: $\angle BAC = 91.5°$

Store the calculator value for A.

To find a second angle it is better to use the sine rule rather than a second cosine rule.

Use $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$, and use the stored value for A.

$\angle ACB = 54.3°$.
$\angle ABC = 180° - (91.5 + 54.3)°$.

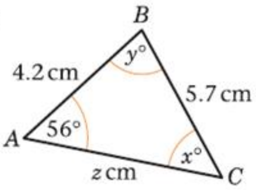
TO DO (Task 7)

Exercise 2F

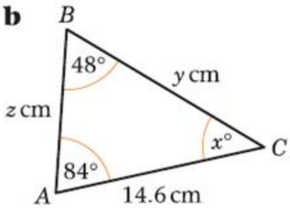
(*Note:* Try to use the neatest method, and give answers to 3 significant figures.)

1 In each triangle below find the values of x, y and z.

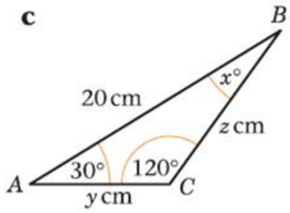
a



b



c



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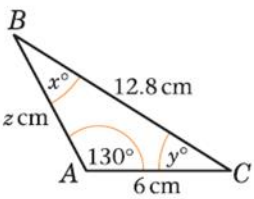
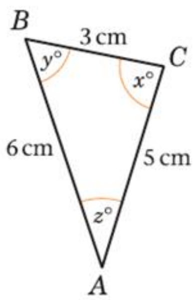


e



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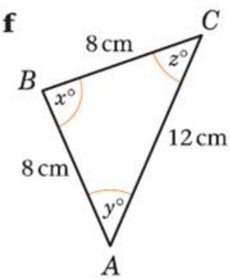
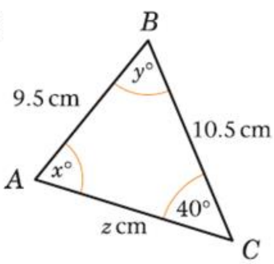
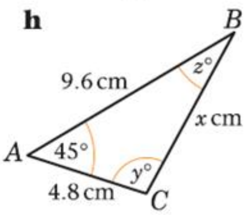


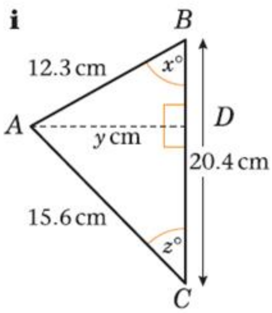
g



h

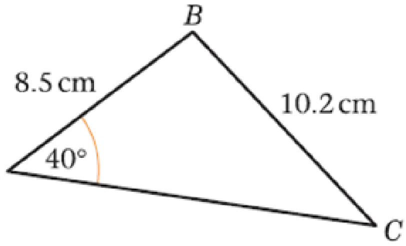


i

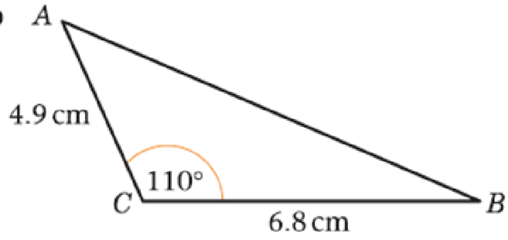


2 Calculate the size of the remaining angles and the length of the third side in the following triangles:

a



b



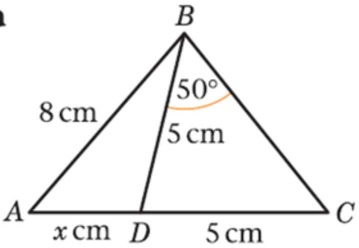
3 A hiker walks due north from A and after 8 km reaches B. She then walks a further 8 km on a bearing of 120° to C. Work out **a** the distance from A to C and **b** the bearing of C from A.

4 A helicopter flies on a bearing of 200° from A to B, where $AB = 70$ km. It then flies on a bearing of 150° from B to C, where C is due south of A. Work out the distance of C from A.

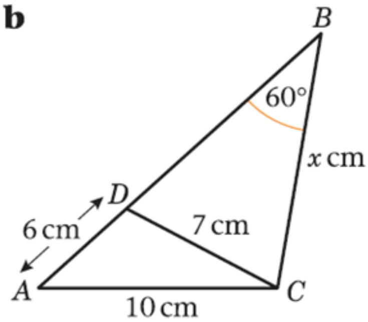
5 Two radar stations A and B are 16 km apart and A is due north of B. A ship is known to be on a bearing of 150° from A and 10 km from B. Show that this information gives two positions for the ship, and calculate the distance between these two positions.

6 Find x in each of the following diagrams:

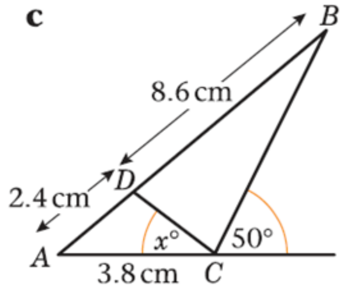
a



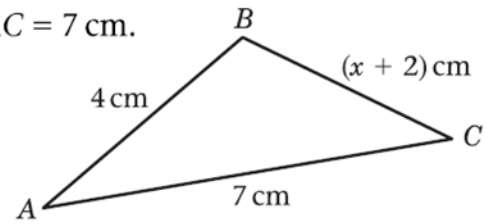
b



c



7 In $\triangle ABC$, shown right, $AB = 4$ cm, $BC = (x + 2)$ cm and $AC = 7$ cm.



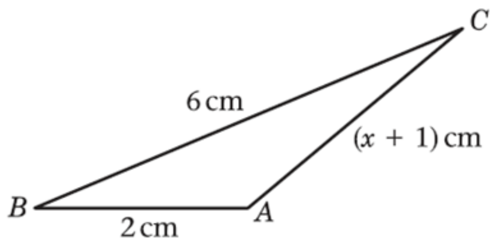
a Explain how you know that $1 < x < 9$.

b Work out the value of x for the cases when

i $\angle ABC = 60°$ and

ii $\angle ABC = 45°$, giving your answers to 3 significant figures.

8 In the triangle shown right, $\cos \angle ABC = \frac{5}{8}$.
Calculate the value of x.



9 In $\triangle ABC$, $AB = \sqrt{2}$ cm, $BC = \sqrt{3}$ cm and $\angle BAC = 60°$. Show that $\angle ACB = 45°$ and find AC.

10 In $\triangle ABC$, $AB = (2 - x)$ cm, $BC = (x + 1)$ cm and $\angle ABC = 120°$:

a Show that $AC^2 = x^2 - x + 7$.

b Find the value of x for which AC has a minimum value.

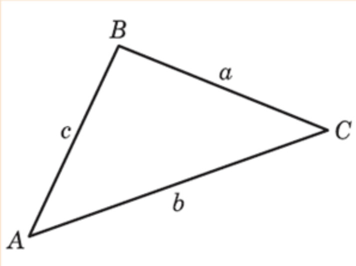
11 Triangle ABC is such that $BC = 5\sqrt{2}$ cm, $\angle ABC = 30°$ and $\angle BAC = \theta$, where $\sin\theta = \dfrac{\sqrt{5}}{8}$.

Work out the length of AC, giving your answer in the form $a\sqrt{b}$, where a and b are integers.

12 The perimeter of $\triangle ABC = 15$ cm. Given that $AB = 7$ cm and $\angle BAC = 60°$, find the lengths of AC and BC.

2.7 You can calculate the area of a triangle using the formula:

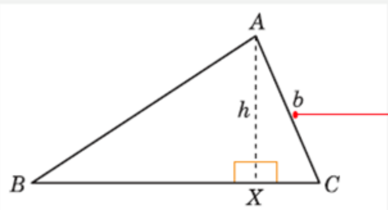
Area of a triangle = $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$.



You can use this formula when you know the lengths of two sides and the size of the angle between them.

Example 15

Show that the area of this triangle is $\frac{1}{2}ab\sin C$.



The perpendicular from A to BC is drawn and it meets BC at X. The length of $AX = h$.

Area of $\triangle ABC = \frac{1}{2}ah$ (1)

Area of triangle $= \frac{1}{2} \times$ base $\times$ height.

But $h = b\sin C$ (2)

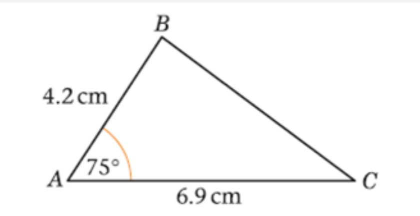
Use the sine ratio $\sin C = \dfrac{h}{b}$ in $\triangle AXC$.

So Area $= \frac{1}{2}ab\sin C$

Substitute 2 into 1.

Example 16

Work out the area of the triangle shown below.



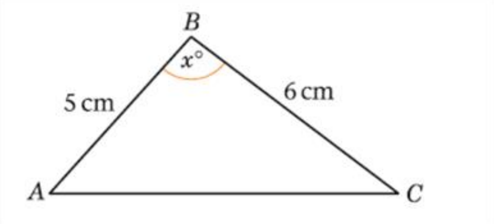
Area of $\triangle ABC = \frac{1}{2} \times 6.9 \times 4.2 \times \sin 75°$ cm^2

$= 14.0$ cm^2 (3 s.f.)

Here $b = 6.9$ cm, $c = 4.2$ cm and angle $A = 75°$, so use:
Area $= \frac{1}{2}bc\sin A$.

Example 17

In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = x°$. Given that the area of $\triangle ABC$ is 12 cm² and that AC is the longest side, find the value of x.



Area $\triangle ABC = \frac{1}{2} \times 5 \times 6 \times \sin x° \text{ cm}^2$

So $12 = \frac{1}{2} \times 5 \times 6 \times \sin x°$

So $\sin x° = 0.8$

$x = 126.9$

$= 127$ (3 s.f.)

Here $a = 6$ cm, $c = 5$ cm and angle $B = x°$, so use:
Area $= \frac{1}{2}ac \sin B$.

Area of $\triangle ABC$ is 12 cm².

$\sin x° = \frac{12}{15}$.

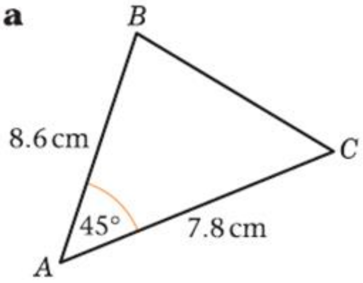
There are two values of x for which $\sin x° = 0.8$, 53.1 and 126.9, but here you know B is the largest angle because AC is the largest side.

TO DO (Task 8)

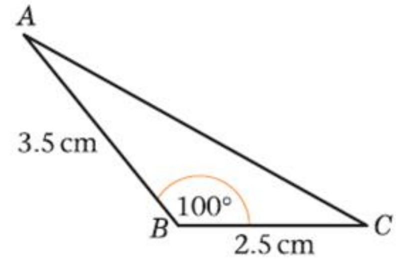
Exercise 2G

1 Calculate the area of the following triangles:

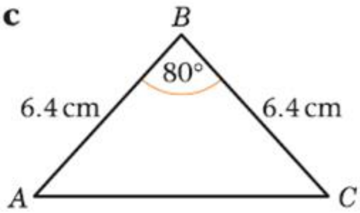
a



b

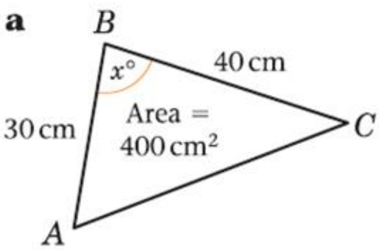


c

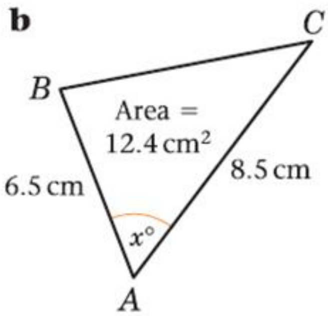


2 Work out the possible values of x in the following triangles:

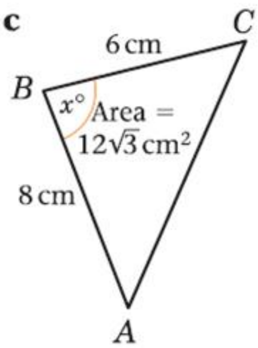
a



b

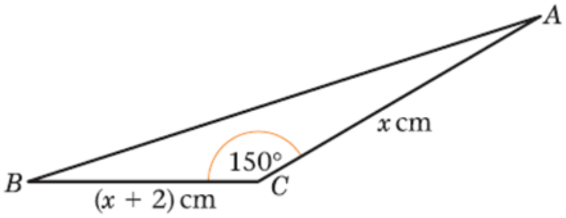


c



3 A fenced triangular plot of ground has area $1200\,m^2$. The fences along the two smaller sides are 60 m and 80 m respectively and the angle between them is $\theta°$. Show that $\theta = 150$, and work out the total length of fencing.

4 In triangle ABC, shown right, $BC = (x + 2)$ cm, $AC = x$ cm and $\angle BCA = 150°$.
Given that the area of the triangle is $5\,cm^2$, work out the value of x, giving your answer to 3 significant figures.



5 In $\triangle PQR$, $PQ = (x + 2)$ cm, $PR = (5 - x)$ cm and $\angle QPR = 30°$.
The area of the triangle is $A\,cm^2$:

a Show that $A = \frac{1}{4}(10 + 3x - x^2)$.

b Use the method of completing the square, or otherwise, to find the maximum value of A, and give the corresponding value of x.

6 In $\triangle ABC$, $AB = x$ cm, $AC = (5 + x)$ cm and $\angle BAC = 150°$. Given that the area of the triangle is $3\frac{3}{4}\,cm^2$:

a Show that x satisfies the equation $x^2 + 5x - 15 = 0$.

b Calculate the value of x, giving your answer to 3 significant figures.

Basic Knowledge of Complex Numbers

1.1 You can use real and imaginary numbers.

When solving a quadratic equation in Unit C1, you saw how the **discriminant** of the equation could be used to find out about the type of roots.

For the equation $ax^2 + bx + c = 0$, the discriminant is $b^2 - 4ac$.
If $b^2 - 4ac > 0$, there are two different real roots.
If $b^2 - 4ac = 0$, there are two equal real roots.
If $b^2 - 4ac < 0$, there are no real roots.

In the case $b^2 - 4ac < 0$, the problem is that you reach a situation where you need to find the square root of a negative number, which is not 'real'.

To solve this problem, another type of number called an 'imaginary number' is used.

The 'imaginary number' $\sqrt{(-1)}$ is called i (or sometimes j in electrical engineering), and sums of real and imaginary numbers, such as $3 + 2\mathrm{i}$, are known as **complex numbers**.

- **A complex number is written in the form $a + b\mathrm{i}$.**
- **You can add and subtract complex numbers.**
- **$\sqrt{(-1)} = \mathrm{i}$**
- **An imaginary number is a number of the form $b\mathrm{i}$, where b is a real number ($b \in \mathbb{R}$).**

Example 1

Write $\sqrt{(-36)}$ in terms of i.

$$\sqrt{(-36)} = \sqrt{(36 \times -1)} = \sqrt{36}\sqrt{(-1)} = 6\mathrm{i}$$

Example 2

Write $\sqrt{(-28)}$ in terms of i.

$$\sqrt{(-28)} = \sqrt{(28 \times -1)} = \sqrt{28}\sqrt{(-1)} = \sqrt{4}\sqrt{7}\sqrt{(-1)} = 2\sqrt{7}\mathrm{i} \text{ or } 2\mathrm{i}\sqrt{7} \text{ or } (2\sqrt{7})\mathrm{i}$$

This can be written as $2\mathrm{i}\sqrt{7}$ or $(2\sqrt{7})\mathrm{i}$ to avoid confusion with $2\sqrt{7\mathrm{i}}$.

Example 3

Solve the equation $x^2 + 9 = 0$.

$$x^2 = -9$$
$$x = \pm\sqrt{(-9)} = \pm\sqrt{(9 \times -1)} = \pm\sqrt{9}\sqrt{(-1)} = \pm 3\mathrm{i}$$
$$x = \pm 3\mathrm{i} \qquad (x = +3\mathrm{i},\ x = -3\mathrm{i})$$

Note that just as $x^2 = 9$ has two roots $+3$ and -3, $x^2 = -9$ also has two roots $+3\mathrm{i}$ and $-3\mathrm{i}$.

- A complex number is a number of the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.
- For the complex number $a + bi$, a is called the **real part** and b is called the **imaginary part.**
- The complete set of complex numbers is called $\mathbb{C}$.

Example 4

Solve the equation $x^2 + 6x + 25 = 0$.

Method 1 (Completing the square)

$x^2 + 6x = (x + 3)^2 - 9$

$x^2 + 6x + 25 = (x + 3)^2 - 9 + 25 = (x + 3)^2 + 16$

$(x + 3)^2 + 16 = 0$

$(x + 3)^2 = -16$

$x + 3 = \pm\sqrt{(-16)} = \pm 4i$

$x = -3 \pm 4i$

$x = -3 + 4i, \quad x = -3 - 4i$

Because
$(x + 3)^2 = (x + 3)(x + 3)$
$= x^2 + 6x + 9$

$\sqrt{(-16)} = \sqrt{(16 \times -1)}$
$= \sqrt{16}\sqrt{(-1)} = 4i$

Method 2 (Quadratic formula)

$$x = \frac{-6 \pm \sqrt{(6^2 - 4 \times 1 \times 25)}}{2} = \frac{-6 \pm \sqrt{(-64)}}{2}$$

$\sqrt{(-64)} = \pm 8i$

$x = \dfrac{-6 \pm 8i}{2} = -3 \pm 4i$

$x = -3 + 4i, \quad x = -3 - 4i$

Using
$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$\sqrt{(-64)} = \sqrt{(64 \times -1)}$
$= \sqrt{64}\sqrt{(-1)} = 8i$

- In a complex number, the real part and the imaginary part cannot be combined to form a single term.
- You can add complex numbers by adding the real parts and adding the imaginary parts.
- You can subtract complex numbers by subtracting the real parts and subtracting the imaginary parts.

Example 5

Simplify, giving your answer in the form $a + bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

a $(2 + 5i) + (7 + 3i)$ **b** $(3 - 4i) + (-5 + 6i)$ **c** $2(5 - 8i)$

d $(1 + 8i) - (6 + i)$ **e** $(2 - 5i) - (5 - 11i)$ **f** $(2 + 3i) - (2 - 3i)$

a	$(2+5i)+(7+3i)=(2+7)+i(5+3)=9+8i$	Add real parts and add imaginary parts.
b	$(3-4i)+(-5+6i)=(3-5)+i(-4+6)=-2+2i$	
c	$2(5-8i)=10-16i$	This is the same as $(5-8i)+(5-8i)$
d	$(1+8i)-(6+i)=(1-6)+i(8-1)=-5+7i$	Subtract real parts and subtract imaginary parts.
e	$(2-5i)-(5-11i)=(2-5)+i(-5-(-11))=-3+6i$	
f	$(2+3i)-(2-3i)=(2-2)+i(3-(-3))=6i$	The answer has no real part. This is called purely imaginary.

TO DO (Task 9):

Exercise 1A

Simplify, giving your answer in the form $a+bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

1 $(5+2i)+(8+9i)$

2 $(4+10i)+(1-8i)$

3 $(7+6i)+(-3-5i)$

4 $(2-i)+(11+2i)$

5 $(3-7i)+(-6+7i)$

6 $(20+12i)-(11+3i)$

7 $(9+6i)-(8+10i)$

8 $(2-i)-(-5+3i)$

9 $(-4-6i)-(-8-8i)$

10 $(-1+5i)-(-1+i)$

11 $(3+4i)+(4+5i)+(5+6i)$

12 $(-2-7i)+(1+3i)-(-12+i)$

13 $(18+5i)-(15-2i)-(3+7i)$

14 $2(7+2i)$

15 $3(8-4i)$

16 $7(1-3i)$

17 $2(3+i)+3(2+i)$

18 $5(4+3i)-4(-1+2i)$

19 $\left(\frac{1}{2}+\frac{1}{3}i\right)+\left(\frac{5}{2}+\frac{5}{3}i\right)$

20 $(3\sqrt{2}+i)-(\sqrt{2}-i)$

Write in the form bi, where $b \in \mathbb{R}$.

21 $\sqrt{(-9)}$

22 $\sqrt{(-49)}$

23 $\sqrt{(-121)}$

24 $\sqrt{(-10\,000)}$

25 $\sqrt{(-225)}$

26 $\sqrt{(-5)}$

27 $\sqrt{(-12)}$

28 $\sqrt{(-45)}$

29 $\sqrt{(-200)}$

30 $\sqrt{(-147)}$

Solve these equations.

31 $x^2 + 2x + 5 = 0$

32 $x^2 - 2x + 10 = 0$

33 $x^2 + 4x + 29 = 0$

34 $x^2 + 10x + 26 = 0$

35 $x^2 - 6x + 18 = 0$

36 $x^2 + 4x + 7 = 0$

37 $x^2 - 6x + 11 = 0$

38 $x^2 - 2x + 25 = 0$

39 $x^2 + 5x + 25 = 0$

40 $x^2 + 3x + 5 = 0$

1.2 You can multiply complex numbers and simplify powers of i.

- **You can multiply complex numbers using the same technique as you use for multiplying brackets in algebra, and you can simplify powers of i.**
- **Since $i = \sqrt{(-1)}$, $i^2 = -1$**

Example 6

Multiply $(2 + 3i)$ by $(4 + 5i)$

$(2 + 3i)(4 + 5i) = 2(4 + 5i) + 3i(4 + 5i)$

Multiply the two brackets as you would in algebra.

$= 8 + 10i + 12i + 15i^2$

$= 8 + 10i + 12i - 15$

Use the fact that $i^2 = -1$.

$= (8 - 15) + (10i + 12i)$

Add real parts and add imaginary parts.

$= -7 + 22i$

Example 7

Express $(7 - 4i)^2$ in the form $a + bi$.

$(7 - 4i)(7 - 4i) = 7(7 - 4i) - 4i(7 - 4i)$

Multiply the two brackets as you would in algebra.

$= 49 - 28i - 28i + 16i^2$

$= 49 - 28i - 28i - 16$

Use the fact that $i^2 = -1$.

$= (49 - 16) + (-28i - 28i)$

Add real parts and add imaginary parts.

$= 33 - 56i$

Example 8

Simplify $(2 - 3i)(4 - 5i)(1 + 3i)$

$(2 - 3i)(4 - 5i) = 2(4 - 5i) - 3i(4 - 5i)$

$= 8 - 10i - 12i + 15i^2 = 8 - 10i - 12i - 15 = -7 - 22i$

$(-7 - 22i)(1 + 3i) = -7(1 + 3i) - 22i(1 + 3i)$

$= -7 - 21i - 22i - 66i^2 = 59 - 43i$

First multiply two of the brackets.

Then multiply the result by the third bracket.

Example 9

Simplify

a i^3 **b** i^4 **c** $(2i)^5$

a $i^3 = i \times i \times i = i^2 \times i = -i$

b $i^4 = i \times i \times i \times i = i^2 \times i^2 = -1 \times -1 = 1$

c $(2i)^5 = 2i \times 2i \times 2i \times 2i \times 2i = 32(i \times i \times i \times i \times i)$

$= 32(i^2 \times i^2 \times i) = 32 \times -1 \times -1 \times i = 32i$

$i^2 = -1$

First multiply the 2s (2^5).

TO DO (Task 10):

Exercise 1B

Simplify these, giving your answer in the form $a + bi$.

1 $(5 + i)(3 + 4i)$

2 $(6 + 3i)(7 + 2i)$

3 $(5 - 2i)(1 + 5i)$

4 $(13 - 3i)(2 - 8i)$

5 $(-3 - i)(4 + 7i)$

6 $(8 + 5i)^2$

7 $(2 - 9i)^2$

8 $(1 + i)(2 + i)(3 + i)$

9 $(3 - 2i)(5 + i)(4 - 2i)$

10 $(2 + 3i)^3$

Simplify.

11 i^6

12 $(3i)^4$

13 $i^5 + i$

14 $(4i)^3 - 4i^3$

15 $(1 + i)^8$

Hint: Use the binomial theorem.

1.3 You can find the complex conjugate of a complex number.

- **You can write down the complex conjugate of a complex number, and you can divide two complex numbers by using the complex conjugate of the denominator.**
- **The complex number $a - bi$ is called the complex conjugate of the complex number $a + bi$.**
- **The complex numbers $a + bi$ and $a - bi$ are called a complex conjugate pair.**
- **The complex conjugate of z is called z^*, so if $z = a + bi$, $z^* = a - bi$.**

Example 10

Write down the complex conjugate of

a $2 + 3i$ **b** $5 - 2i$ **c** $\sqrt{3} + i$ **d** $1 - i\sqrt{5}$

a $2 - 3i$ **b** $5 + 2i$

c $\sqrt{3} - i$ **d** $1 + i\sqrt{5}$

Just change the sign of the imaginary part (from + to −, or − to +).

Example 11

Find $z + z^*$ and zz^*, given that

a $z = 3 + 5i$ **b** $z = 2 - 7i$ **c** $z = 2\sqrt{2} + i\sqrt{2}$

a $z^* = 3 - 5i$

$z + z^* = (3 + 5i) + (3 - 5i) = (3 + 3) + i(5 - 5) = 6$ — Note that $z + z^*$ is real.

$zz^* = (3 + 5i)(3 - 5i) = 3(3 - 5i) + 5i(3 - 5i)$

$= 9 - 15i + 15i - 25i^2 = 9 + 25 = 34$ — Note that zz^* is real.

b $z^* = 2 + 7i$

$z + z^* = (2 - 7i) + (2 + 7i) = (2 + 2) + i(-7 + 7) = 4$ — Note that $z + z^*$ is real.

$zz^* = (2 - 7i)(2 + 7i) = 2(2 + 7i) - 7i(2 + 7i)$

$= 4 + 14i - 14i - 49i^2 = 4 + 49 = 53$ — Note that zz^* is real.

c $z^* = 2\sqrt{2} - i\sqrt{2}$

$z + z^* = (2\sqrt{2} + i\sqrt{2}) + (2\sqrt{2} - i\sqrt{2})$

$= (2\sqrt{2} + 2\sqrt{2}) + i(\sqrt{2} - \sqrt{2}) = 4\sqrt{2}$ — Note that $z + z^*$ is real.

$zz^* = (2\sqrt{2} + i\sqrt{2})(2\sqrt{2} - i\sqrt{2})$

$= 2\sqrt{2}(2\sqrt{2} - i\sqrt{2}) + i\sqrt{2}(2\sqrt{2} - i\sqrt{2})$

$= 8 - 4i + 4i - 2i^2 = 8 + 2 = 10$ — Note that zz^* is real.

Example 12

Simplify $(10 + 5i) \div (1 + 2i)$

$$(10 + 5i) \div (1 + 2i) = \frac{10 + 5i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i}$$

The complex conjugate of the denominator is $1 - 2i$. Multiply numerator and denominator by this.

$$\frac{10 + 5i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} = \frac{(10 + 5i)(1 - 2i)}{(1 + 2i)(1 - 2i)}$$

$$\begin{aligned}(10 + 5i)(1 - 2i) &= 10(1 - 2i) + 5i(1 - 2i)\\ &= 10 - 20i + 5i - 10i^2\\ &= 20 - 15i\end{aligned}$$

$$\begin{aligned}(1 + 2i)(1 - 2i) &= 1(1 - 2i) + 2i(1 - 2i)\\ &= 1 - 2i + 2i - 4i^2 = 5\end{aligned}$$

$$(10 + 5i) \div (1 + 2i) = \frac{20 - 15i}{5} = 4 - 3i$$

Divide each term in the numerator by 5.

Example 13

Simplify $(5 + 4i) \div (2 - 3i)$

$$(5 + 4i) \div (2 - 3i) = \frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$

The complex conjugate of the denominator is $2 + 3i$. Multiply numerator and denominator by this.

$$\frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i} = \frac{(5 + 4i)(2 + 3i)}{(2 - 3i)(2 + 3i)}$$

$$\begin{aligned}(5 + 4i)(2 + 3i) &= 5(2 + 3i) + 4i(2 + 3i)\\ &= 10 + 15i + 8i + 12i^2\\ &= -2 + 23i\end{aligned}$$

$$\begin{aligned}(2 - 3i)(2 + 3i) &= 2(2 + 3i) - 3i(2 + 3i)\\ &= 4 + 6i - 6i - 9i^2 = 13\end{aligned}$$

$$(5 + 4i) \div (2 - 3i) = \frac{-2 + 23i}{13} = -\frac{2}{13} + \frac{23}{13}i$$

Divide each term in the numerator by 13.

The division process shown in Examples 12 and 13 is similar to the process used to divide surds. (See C1 Section 1.8.)
For surds the denominator is rationalised. For complex numbers the denominator is made real.

- **If the roots α and β of a quadratic equation are complex, α and β will always be a complex conjugate pair.**

- **If the roots of the equation are α and β, the equation is $(x - \alpha)(x - \beta) = 0$**
  $(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 - (\alpha + \beta)x + \alpha\beta$

Example 14

Find the quadratic equation that has roots $3 + 5i$ and $3 - 5i$.

For this equation $\alpha + \beta = (3 + 5i) + (3 - 5i) = 6$
and $\alpha\beta = (3 + 5i)(3 - 5i) = 9 + 15i - 15i - 25i^2 = 34$
The equation is $x^2 - 6x + 34 = 0$

TO DO (Task 11):

Exercise 1C

1 Write down the complex conjugate z^* for

a $z = 8 + 2i$
b $z = 6 - 5i$
c $z = \frac{2}{3} - \frac{1}{2}i$
d $z = \sqrt{5} + i\sqrt{10}$

2 Find $z + z^*$ and zz^* for

a $z = 6 - 3i$
b $z = 10 + 5i$
c $z = \frac{3}{4} + \frac{1}{4}i$
d $z = \sqrt{5} - 3i\sqrt{5}$

Find these in the form $a + bi$.

3 $(25 - 10i) \div (1 - 2i)$

4 $(6 + i) \div (3 + 4i)$

5 $(11 + 4i) \div (3 + i)$

6 $\dfrac{1 + i}{2 + i}$

7 $\dfrac{3 - 5i}{1 + 3i}$

8 $\dfrac{3 + 5i}{6 - 8i}$

9 $\dfrac{28 - 3i}{1 - i}$

10 $\dfrac{2 + i}{1 + 4i}$

11 $\dfrac{(3 - 4i)^2}{1 + i}$

Given that $z_1 = 1 + i$, $z_2 = 2 + i$ and $z_3 = 3 + i$, find answers for questions 12–14 in the form $a + bi$.

12 $\dfrac{z_1 z_2}{z_3}$

13 $\dfrac{(z_2)^2}{z_1}$

14 $\dfrac{2z_1 + 5z_3}{z_2}$

15 Given that $\dfrac{5 + 2i}{z} = 2 - i$, find z in the form $a + bi$.

16 Simplify $\dfrac{6 + 8i}{1 + i} + \dfrac{6 + 8i}{1 - i}$, giving your answer in the form $a + bi$.

17 The roots of the quadratic equation $x^2 + 2x + 26 = 0$ are α and β.
Find **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

18 The roots of the quadratic equation $x^2 - 8x + 25 = 0$ are α and β.
Find **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

19 Find the quadratic equation that has roots $2 + 3\text{i}$ and $2 - 3\text{i}$.

20 Find the quadratic equation that has roots $-5 + 4\text{i}$ and $-5 - 4\text{i}$.

CHALLENGE:

1. The complex number z is defined by $z = \dfrac{8 + p\text{i}}{p - 4\text{i}}$, $p \in \mathbb{R}$. Given that the real part of z is $\dfrac{2}{5}$,

a find the possible values of p. **(4)**

b Write the possible values of z in the form $a + bi$, where a and b are real. **(1)**

2. $z = \dfrac{4}{1 + \text{i}}$

a Find z in the form $a + bi$, where a and b are real. **(2)**

b Given that z is a complex root of the quadratic equation $px^2 + qx + r = 0$, where p, q and r are integers find possible values of p, q and r. **(4)**

3. Given that $z = x + \text{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$, find the value of x and the value of y such that

$$(3 - \text{i})z^* + 2\text{i}z = 9 - \text{i}$$

where z^* is the complex conjugate of z.

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